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British Mathematical Olympiad

Round 2 : Tuesday, 25 February 2003

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions 1,2,3,4 in order, and the cover sheet at the front.*

In early March, twenty students will be invited to attend the training session to be held at Trinity College, Cambridge (3-6 April). On the final morning of the training session, students sit a paper with just 3 Olympiad-style problems, and 8 students will be selected for further training. Those selected will be expected to participate in correspondence work and to attend further training. The UK Team of 6 for this summer's International Mathematical Olympiad (to be held in Japan, 7-19 July) will then be chosen.

Do not turn over until **told to do so.**

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2003 British Mathematical Olympiad
Round 2

1. For each integer $n > 1$, let $p(n)$ denote the largest prime factor of n.
   Determine all triples x, y, z of distinct positive integers satisfying
   (i) x, y, z are in arithmetic progression, and
   (ii) $p(xyz) \leq 3$.

2. Let ABC be a triangle and let D be a point on AB such that $4AD = AB$. The half-line ℓ is drawn on the same side of AB as C, starting from D and making an angle of θ with DA where $\theta = \angle ACB$. If the circumcircle of ABC meets the half-line ℓ at P, show that $PB = 2PD$.

3. Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of the set $\mathbb{N}$ of all positive integers.
   (i) Show that there is an arithmetic progression of positive integers $a, a+d, a+2d$, where $d > 0$, such that
   $$f(a) < f(a+d) < f(a+2d).$$
   (ii) Must there be an arithmetic progression $a, a+d, \ldots, a+2003d$, where $d > 0$, such that
   $$f(a) < f(a+d) < \ldots < f(a+2003d)?$$
   [*A permutation of* $\mathbb{N}$ *is a one-to-one function whose image is the whole of* $\mathbb{N}$*; that is, a function from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that for all* $m \in \mathbb{N}$ *there exists a unique* $n \in \mathbb{N}$ *such that* $f(n) = m$.]

4. Let f be a function from the set of non-negative integers into itself such that for all $n \geq 0$
   (i) $\big(f(2n+1)\big)^2 - \big(f(2n)\big)^2 = 6f(n) + 1$, and
   (ii) $f(2n) \geq f(n)$.
   How many numbers less than 2003 are there in the image of f?